The Gamma Function

Motivation

For integer n, $n! = n \cdot (n-1) \cdots 1$

How about $\left(\frac{7}{2}!\right)$? Does it make sense to talk about $\left(\frac{7}{2}!\right)$?

(a) $\Gamma(x)$: The Gamma Function

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z}\, dz , \qquad x > 0 \qquad (1)$$

Note: z is integrated over, what is left is x

$\therefore$ $\boxed{\text{It is a function of } x.}$

Ex: $\Gamma(1) = \int_0^\infty e^{-z} dz = 1$ (the first integrals you learned in calculus)

$\Gamma(2) = \int_0^\infty z e^{-z} dz = \left(-z e^{-z}\right)\Big|_0^\infty + \int_0^\infty e^{-z} dz$ (by parts) $= 1$

Consider x being integers ≥ 2

$$\Gamma(x) = \int_0^{\infty} z^{x-1} e^{-z} dz = -\int_0^{\infty} z^{x-1} d(e^{-z}) \overset{\text{by parts}}{=} \cancel{\left[-z^{x-1} e^{-z}\right]_0^{\infty}}^{\to 0} + (x-1) \overbrace{\underbrace{\int_0^{\infty} z^{x-2} e^{-z} dx}_{\text{OK } (x \geq 2)}}^{\Gamma(x-1)}$$

$$\therefore \boxed{\Gamma(x) = (x-1)\Gamma(x-1)} \qquad (2)$$

e.g.

$$\Gamma(5) = 4\,\Gamma(4) = 4\cdot 3\cdot \Gamma(3) = 4\cdot 3\cdot 2\cdot \Gamma(2) = 4\cdot 3\cdot 2\cdot 1\cdot \overbrace{\Gamma(1)}^{1} = 4!$$

$$\therefore \Gamma(5) = 4!$$

$$\boxed{\Gamma(x) = (x-1)! \quad \text{for } x = 2, 3, 4, \cdots} \qquad (3)$$

By product: $\Gamma(1) = 1 = 0!$ (We don't need $0!$ in Stat. Mech. We need $10^{23}!$)

$\therefore$ $\Gamma(x)$ Gamma Function is a way to represent $n!$ for positive integers.

Further Motivation

Single-particle partition function in classical ideal gas

$$z = \sum_{\text{all s.p. states } i} e^{-\varepsilon_i/kT} = \int g_{3D}(\varepsilon)\, e^{-\varepsilon/kT} d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/kT} d\varepsilon}_{\text{relates to } \Gamma\left(\frac{3}{2}\right)}$$

- In Statistical Mechanics, we also need $\Gamma\left(\frac{3}{2}\right)$, $\Gamma\left(\frac{5}{2}\right)$, $\Gamma(1/2)$ or $\Gamma(\text{half-integer})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty z^{-1/2} e^{-z} dz \qquad \text{Let } u^2 = z \Rightarrow z^{1/2} = u, \; dz = 2u\,du$$

$$= \int_0^\infty \frac{1}{u} e^{-u^2} \cdot 2u\,du$$

$$= 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du = \sqrt{\pi} \quad \text{(Gaussian Integral)} \qquad (4)$$

- Every step in getting $\Gamma(x) = (x-1)\Gamma(x-1)$ is still valid

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \quad \left(\text{this is what } \tfrac{1}{2}! \text{ meant!}\right) \quad (5)$$

Immediate Application:

$$z = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/kT} d\varepsilon$$

let $z = \varepsilon/kT$, $d\varepsilon = kT\, dz$, $\varepsilon^{1/2} = z^{1/2}(kT)^{1/2}$

$$= \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(kT)^{3/2} \underbrace{\int_0^\infty z^{1/2} e^{-z} dz}_{\Gamma\left(\frac{3}{2}\right)} = \frac{V}{4\pi^2}\frac{(2mkT)^{3/2}}{\frac{h^3}{8\pi^3}} \cdot \frac{\sqrt{\pi}}{2} = \frac{V}{\left(\frac{h}{\sqrt{2\pi mkT}}\right)^3} = \frac{V}{\lambda_{th}^3}$$

(an old result)

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi} \quad (6)$$

(this is $\underbrace{\tfrac{3}{2}!}_{\text{"factorial of } 3/2\text{"}}$)

Remark:

$\Gamma(x)$ also works for other values of x, including negative x (except $x = -1, -2, \cdots$), so we can find $(-\tfrac{1}{2}!)$, but such $\Gamma(x)$ don't concern us in stat. mech.

The Stirling Approximation

$\Gamma(x+1) = x\,\Gamma(x) = x!$ and it is an integral

$$\Gamma(x+1) = \int_0^\infty z^x e^{-z}\, dz$$

$\boxed{\text{Q: Any good approximation for large } x\text{?}}$ †

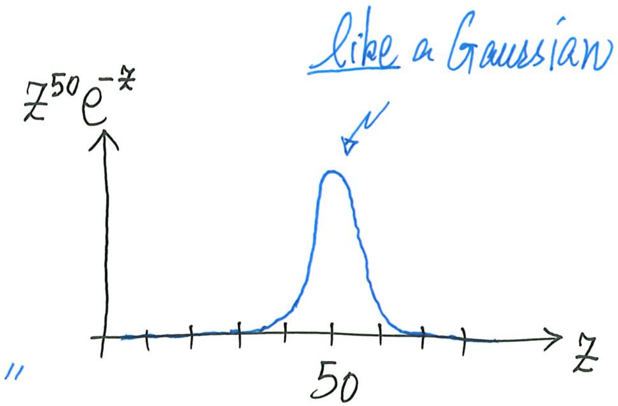
Look at Integrand : $z^x e^{-z}$ as a function of z for a given x (e.g. $x=50$)

Where does $z^x e^{-z}$ peak?

$$\frac{d}{dz}\left(z^x e^{-z}\right) = x z^{x-1} e^{-z} - z^x e^{-z}$$

Peaks at $\frac{d}{dz}(z^x e^{-z}) = 0 \Rightarrow \underbrace{z = x}$

"does it look obvious to you?"



† The math steps in this derivation are useful in many other contexts, e.g. in mean field theories in stat. mech. problems.

Since $z^x = e^{\ln z^x} = e^{x \ln z}$, $\Gamma(x+1) = \int_0^\infty \underbrace{e^{x \ln z - z}}_{\text{peaks at } z = x \text{ (bigger } x\text{, sharper)}} dz = x!$ (7)

$\boxed{\text{Integral is dominated by } e^{x \ln z - z} \text{ for } z \approx x}$ ← idea for making approximation

Also, e^y is a monotonic function of y (y goes up(down), e^y goes up(down))
i.e. $(x \ln z - z)$ also peaks at $z = x$

Approximation Here! (Extract behavior of $(x \ln z - z)$ NEAR $z = x$)

$$x \ln z - z \approx (x \ln x - x) + \cancel{\frac{d}{dz}(x \ln z - z)\Big|_{z=x}}^{\to 0} \cdot (z - x) + \frac{1}{2}\left(\frac{d^2}{dz^2}(x \ln z - z)\right)_{z=x} \cdot (z-x)^2 + \underbrace{\cdots}_{\text{neglect}}$$

(∵ $z = x$ is a peak)

$$\approx (x \ln x - x) - \frac{1}{2x} \cdot (z-x)^2 \quad (8)$$

(Key step)

$$\Gamma(x+1) = \int_0^\infty e^{x\ln z - z}\,dz \cong e^{x\ln x - x}\underbrace{\int_0^\infty e^{-\frac{(z-x)^2}{2x}}\,dz} = x^x e^{-x}\int_0^\infty e^{-\frac{(z-x)^2}{2x}}\,dz$$

← started[+] to look like a Gaussian integral

Call $\frac{(z-x)^2}{2x} = y^2$, $dz = \sqrt{2x}\,dy$

$$\therefore \Gamma(x+1) \cong x^x e^{-x}\int_{-\sqrt{\frac{x}{2}}}^\infty e^{-y^2}\,dy\cdot\sqrt{2x}$$

We want to have an approximation of $x!$ for large x, so lower limit $\approx \infty$

$$\Gamma(x+1) = x! \approx x^x e^{-x}\sqrt{2x}\cdot\sqrt{\pi} = \sqrt{2\pi x}\cdot x^x e^{-x}$$

$$\boxed{\begin{aligned} x! &\approx x^x e^{-x}\sqrt{2\pi x} \quad (9)\ \text{(Stirling Approximation)} \\ \ln x! &\approx x\ln x - x + \tfrac{1}{2}\ln(2\pi x) \\ &\approx x\ln x - x \quad (10)\ \text{(Stirling Approximation)} \end{aligned}}$$

[+] This technique is often called making the Gaussian Approximation and the Saddle-Point Method.

Summary

$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz$ generalizes $x!$ to factorials of non-integers

$\Gamma\left(\frac{5}{2}\right)$, $\Gamma\left(\frac{3}{2}\right)$, $\Gamma(1/2)$ are useful in stat. mech. calculations

$\Gamma(x)$ gives a formal proof of $\ln x! \approx x \ln x - x$ (Stirling Approximation)

References

McQuarrie, "Mathematical Methods for Scientists and Engineers", Ch. 3

Mathews and Walker, "Mathematical Methods of Physics", Ch. 3